Lifting discrete trajectories

Vakhtang Lomadze*

The starting point of this note has been E. Borel's theorem, which states that every formal series can be represented as the Taylor expansion of some C^∞-function. This nice fact, gives rise to the following natural question: Can any discrete trajectory of a polynomial matrix be represented as the Taylor expansion of a continuous one?

More precisely. Let $\mathbb{F}$ be either $\mathbb{R}$ or $\mathbb{C}$, s an indeterminate and $t = s^{-1}$. Let I be an interval containing 0, $\partial : C^\infty(I) \to C^\infty(I)$ the differentiation operator, $\sigma : \mathbb{F}[[t]] \to \mathbb{F}[[t]]$ the backward shift operator.

Define the operator $T : C^\infty(I) \to \mathbb{F}[[t]]$ by the formula

$$T(w) = w(0) + w'(0)t + w''(0)t^2 + \cdots .$$

This is surjective by E. Borel's theorem. Remark that $T \circ \partial = \sigma \circ T$.

Now, let $R \in \mathbb{F}[s]^{\bullet\times\bullet}$. In view of the above remark, we clearly have $T \circ R(\partial) = R(\sigma) \circ T$. It is immediate from this that T induces a map

$$KerR(\partial) \to KerR(\sigma).$$

In other words, T transforms continuous trajectories of R into discrete trajectories of R. The question is whether this map is surjective.

We shall prove that the map is surjective; even more, we shall find its kernel.

*Mathematics department, Ivane Javakhishvili Tbilisi State University (vakhtang.lomadze@tsu.ge)